Tutorial 5 (14 Oct)

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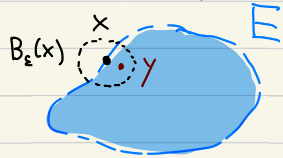
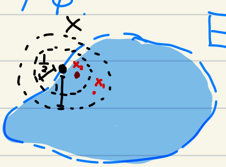
Q1) Given a non-empty subset E of a metric space (X,d), a point $x \in X$ is a limit point of E if every open ball centered at x contains a point $y \in E$ with $y \neq x$, i.e. $\forall \varepsilon > 0, (B_\varepsilon(x) \setminus \{x\}) \cap E \neq \phi$.



Show that x is a limit point of E if and only if $x = \lim_{n\to\infty} x_n$ for some $x_n \in E$ with $x_n \neq x$, $\forall n \in \mathbb{N}$.

Sol) Idea: Construct a sequence by setting $\varepsilon = \frac{1}{n}$ for each $n \in \mathbb{N}$.

[$\Rightarrow$] For each $n \in \mathbb{N}$, choose $\varepsilon = \frac{1}{n} > 0$. Then $(B_{\frac{1}{n}}(x) \setminus \{x\}) \cap E \neq \phi$.



Choose any $x_n \in (B_{\frac{1}{n}}(x) \setminus \{x\}) \cap E$, hence $x_n \neq x$ and $x_n \in E$.

Showing $\lim_{n\to\infty} x_n = x$: $\forall \varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.

Then $\forall n \geq N$, $d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. $\therefore \lim_{n\to\infty} x_n = x$.

[$\Leftarrow$] $\forall \varepsilon > 0$, as $\lim_{n\to\infty} x_n = x$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $d(x_n, x) < \varepsilon$.

In particular, $x_N \in (B_\varepsilon(x) \setminus \{x\}) \cap E$, hence $(B_\varepsilon(x) \setminus \{x\}) \cap E \neq \phi$.

Therefore, x is a limit point of E

Rmk · This question justifies the term "limit point":

a limit point x is the limit of a sequence of points (distinct from x).

- It is closely related to the concept of closure as demonstrated in the next problem.

Q2) Given a non-empty subset E of a metric space (X,d), its derived set $E' \subseteq X$ is the set of limit points of E in X.

(a) Show that $\overline{E} = E \cup E'$.

(b) Hence, show that when $E = [a,b) \subseteq (\mathbb{R}, |\cdot|)$, then $E' = [a,b]$.

Sol) (a) Idea: Use an equivalent definition of $\overline{E}$ as in Prop. 2.9(a) from lecture.

Note that $x \in \overline{E} \iff \forall \varepsilon > 0,\ B_\varepsilon(x) \cap E \neq \phi$ (by Prop. 2.9(a))

$\iff \forall \varepsilon > 0,\ (B_\varepsilon(x) \setminus \{x\}) \cap E \neq \phi$ or $\{x\} \cap E \neq \phi$

$\iff x \in E'$ or $x \in E \iff x \in E \cup E'$.

(b) Idea: Use (a) for [$\subseteq$] and the definition of limit point for [$\supseteq$].

- $E' \subseteq [a,b]$: Note that $\overline{E} = E \cup \partial E = [a,b]$. $\therefore$ By (a), $E' \subseteq [a,b]$.
- $[a,b] \subseteq E'$: Case 1: $x \in [a,b)$: Note that $x = \lim_{n\to\infty} \left(x + \frac{1}{n+N}\right)$,

where $N \in \mathbb{N}$ such that $x + \frac{1}{N} < b$. Hence $x_n = x + \frac{1}{n+N} \in E$. $\therefore$ by (Q1a), $x \in E'$.

Case 2: $x = b$: Note that $b = \lim_{n\to\infty} \left(b - \frac{1}{n+N}\right)$, where $N \in \mathbb{N}$ such that $b - \frac{1}{N} > a$.

Hence $x_n = b - \frac{1}{n+N} \in E$. $\therefore$ by (Q1a), $b \in E'$.

Rmk. (a) shows $\overline{E} = E \cup E'$; meanwhile $\overline{E} := E \cup \partial E$ by lecture note.

Hence, E is closed $\Leftrightarrow \overline{E} = E \Leftrightarrow E \supseteq E' \Leftrightarrow E \supseteq \partial E$.

One might wonder how E' and ∂E are related.

- However, (b) shows that $E' \not\subseteq \partial E$ in general: $([a,b))' = [a,b]$ and $\partial [a,b) = \{a,b\}$.
- Q3b will show that $E' \not\supseteq \partial E$ in general.
- Hence, E' and ∂E are not "subconcept" of each other.

Q3) Given a non-empty subset E of a metric space (X,d),

(a) Given $x \in E'$, show that for any $\varepsilon > 0$, $B_\varepsilon(x) \cap E$ is an infinite set.

(b) Hence, show that when $E = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq (\mathbb{R}, |\cdot|)$, then $E' = \{0\}$.

Sol) (a) Idea: Prove by contradiction, using the assumption of limit point.

Suppose on the contrary, then there exists $\varepsilon_0 > 0$ such that

$(B_{\varepsilon_0}(x) \setminus \{x\}) \cap E = \{x_1, \cdots, x_n\}$ is finite. Choose $0 < \varepsilon < \min\limits_{1 \le i \le n} \{d(x, x_i)\} < \varepsilon_0$.

Since x is a limit point, $(B_\varepsilon(x) \setminus \{x\}) \cap E \neq \phi$. Choose any $y \in (B_\varepsilon(x) \setminus \{x\}) \cap E$,

as $\varepsilon < \varepsilon_0$, $y \in (B_{\varepsilon_0}(x) \setminus \{x\}) \cap E$, hence $y = x_i$, for some $1 \le i \le n$. $\therefore d(x, y_i) > \varepsilon$.

However, $d(x, y) < \varepsilon$ by assumption. This is a contradiction.

$\therefore$ for any $x \in E'$, $\varepsilon > 0$, $B_\varepsilon(x) \cap E$ is an infinite set.

(b) Idea: Checking whether a given element is a limit point by Q1a and Q3a.

[$\supseteq$] Since $\lim\limits_{n \to \infty} \frac{1}{n} = 0$, by (Q1a), 0 is a limit point of E.

[$\subseteq$] Equivalently, showing $\forall x \neq 0, x \notin E'$: Choose $\varepsilon = \frac{|x|}{2} > 0$, then

$$B_\varepsilon(x) \subseteq \begin{cases} (\frac{x}{2}, +\infty), & x > 0 \\ (-\infty, \frac{x}{2}), & x < 0 \end{cases}$$

, hence $B_\varepsilon(x) \cap E$ is finite (or empty).

$\therefore$ By (a), $x \notin E'$. $\therefore E' = \{0\}$.

<u>Rmk</u> • (a) is not true if E' is replaced by $\overline{E}$:

for example, $E = \{x\} \subseteq (X, d)$, then $x \in \overline{E}$, but $B_\varepsilon(x) \cap E = \{x\}$.

• (b) show that $E' \neq \partial E$ in general: $\partial E = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

• (b) generalises to the following setting: $E = \{x_n \mid n \in \mathbb{N}\} \subseteq (X, d)$,

where (x_n) is a sequence of distinct elements with $\lim_{n\to\infty} x_n = y$. Then $E' = \{y\}$.